

3-4 Complex eigenvalues

Example: $\vec{x}' = A\vec{x} = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \vec{x}$ (1)

eigenvalues: $\lambda_1 = -2+i, \lambda_2 = -2-i$

eigenvectors: $\vec{v}_1 = \begin{pmatrix} 2 \\ i-1 \end{pmatrix}, \vec{v}_2 = \vec{v}_1^* = \begin{pmatrix} 2 \\ -i+1 \end{pmatrix}$

general solution: $\vec{x}(t) = c_1 e^{-2t} \vec{v}_1 + c_2 e^{-2t} \vec{v}_2$
 $= c_1 e^{-2t+i} \begin{pmatrix} 2 \\ i-1 \end{pmatrix} + c_2 e^{-2t-i} \begin{pmatrix} 2 \\ -i+1 \end{pmatrix}$

Pb: we want real valued linearly independent solutions to (1).

Idea: use the real part and the imaginary part of the solution.

Set $w_1 = e^{-2+i} \begin{pmatrix} 2 \\ i-1 \end{pmatrix} = e^{-2t} \vec{v}_1$ (for us: $\vec{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

Set $\lambda_1 = \alpha + i\beta, \vec{v}_1 = \vec{a} + i\vec{b}$

$\vec{w}_1 = e^{-2t} e^{i\beta t} (\vec{a} + i\vec{b}) \quad \rightarrow$ Euler formula

$= e^{-2t} (\cos(\beta t) + i \sin(\beta t)) (\vec{a} + i\vec{b})$

$= \boxed{e^{-2t} (\cos(\beta t) \vec{a} - \sin(\beta t) \vec{b}) + i e^{-2t} (\sin(\beta t) \vec{a} + \cos(\beta t) \vec{b})}$

$\vec{w}_1 = \vec{x}_1 + i \vec{x}_2$

Fact: \vec{x}_1 and \vec{x}_2 are two linearly independent solutions to (1).

~~the linear combination of \vec{x}_1 and \vec{x}_2 is also a solution to (1)~~

General form of the real solutions:

$\boxed{\vec{x}(t) = C_1 e^{-2t} (\cos(\beta t) \vec{a} - \sin(\beta t) \vec{b}) + C_2 e^{-2t} (\sin(\beta t) \vec{a} + \cos(\beta t) \vec{b})}$

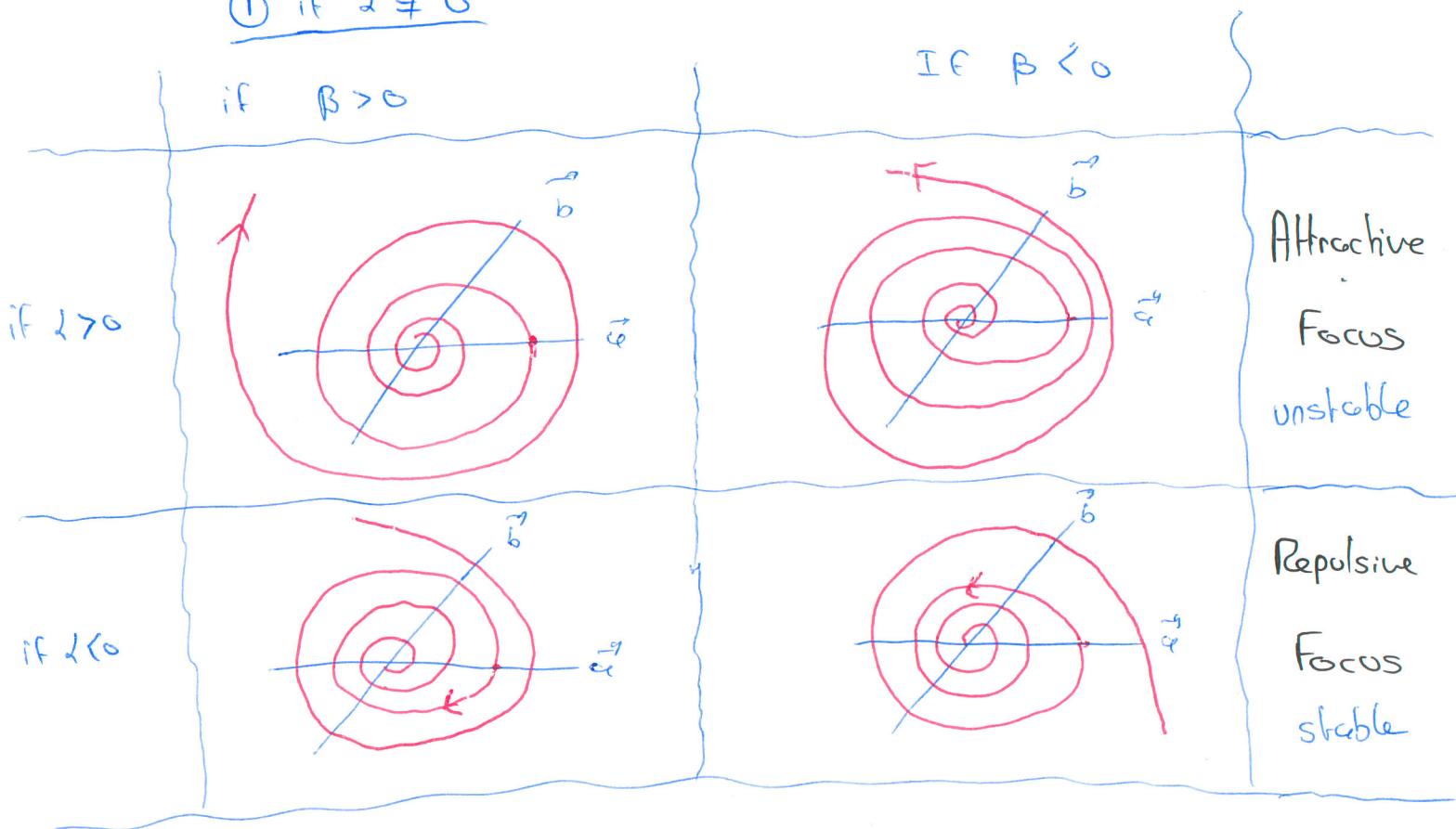
To memorize.

Phase portraits:

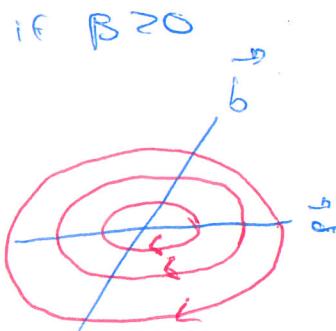
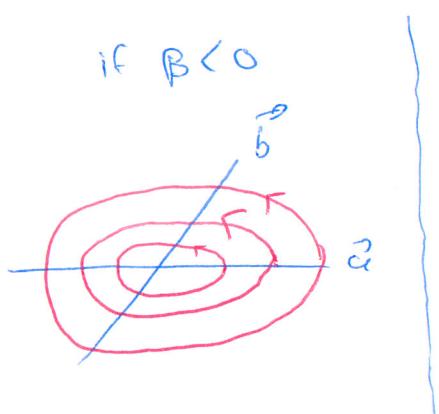
$$\vec{x}(t) = c_1 e^{\lambda t} \underbrace{\left(\cos(\beta t) \vec{a} - \sin(\beta t) \vec{b} \right)}_{\text{periodic}} + c_2 e^{2t} \underbrace{\left(\sin(2t) \vec{a} + \cos(2t) \vec{b} \right)}_{\text{periodic}}$$

" $\vec{x}(t) = e^{\lambda t} \times \text{periodic function}"$

① if $\lambda \neq 0$



② if $\lambda = 0$



: Center
stable
but not asymptotically stable.